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STAR COMPLEMENTS AND EDGE-CONNECTIVITY IN FINITE GRAPHS

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Abstract

Let G be a finite graph with H as a star complement for a non-zero eigenvalue μ . Let $\kappa'(G)$, $\delta(G)$ denote respectively the edge-connectivity and minimum degree of G . We show that $\kappa'(G)$ is controlled by $\delta(G)$ and $\kappa'(H)$. We describe the possibilities for a minimum cutset of G when $\mu \notin \{-1, 0\}$. For such μ , we establish a relation between $\kappa'(G)$ and the spectrum of H when G has a non-trivial minimum cutset $E \not\subseteq E(H)$.

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1 Introduction

Let G be a finite simple graph with μ as an eigenvalue of multiplicity k . (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0, 1)$ -adjacency matrix A of G has dimension k .) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G . We use the notation of [8], where the basic properties of star sets and star complements are established in Chapter 5.

If G has H as a star complement of order t , for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$ [2, Theorem 2.3]. Thus there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1 . In these circumstances, it is of interest to investigate properties of H that are reflected in G : connectedness is one such property, as observed in [11, Section 2]. It was shown in [13] that the vertex-connectivity $\kappa(G)$ is controlled by $\kappa(H)$ and the minimum degree $\delta(G)$. In particular, for each $k \in \mathbb{N}$, there exists a smallest non-negative integer $f(k)$ such that

$$\mu \notin \{-1, 0\}, \kappa(H) \geq k, \delta(G) \geq f(k) \Rightarrow \kappa(G) \geq k.$$

Here we first establish an analogous result for edge-connectivity: for each $k \in \mathbb{N}$, there exists a smallest non-negative integer $g(k)$ such that

$$\mu \neq 0, \kappa'(H) \geq k, \delta(G) \geq g(k) \Rightarrow \kappa'(G) \geq k. \quad (1)$$

The arguments for $\kappa'(G)$ are quite different from those for $\kappa(G)$, and rely on a property of dominating sets. Moreover, whereas little is known about the function f , we find that $g(1) = 0$ and $g(k) = k$ for all $k > 1$. (It was shown in [13] that $k \leq f(k) \leq \frac{1}{2}(k-1)(k+2)$, while $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $f(6) \geq 8$.)

We go on to investigate the nature of minimum cutsets of G when $\mu \notin \{-1, 0\}$. Following [9], we say that such a cutset E is *trivial* if E consists of the edges containing a vertex v (necessarily of degree $\delta(G)$). The interesting case is that in which G has a nontrivial minimum cutset E not in $E(H)$, for then we can find an upper bound for $\kappa'(G)$ in terms of the spectrum of H . We note some consequences in the case that H is regular and μ is not a main eigenvalue.

2 Preliminaries

We take $V(G) = \{1, \dots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. The eigenvalues of G are denoted by $\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)$, in non-increasing order. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S , and $\Delta_S(u)$ for the S -neighbourhood $\{v \in S : v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. An all-1 vector is denoted by \mathbf{j} , its length determined by context.

The following result, known as the Reconstruction Theorem, is fundamental to the theory of star complements.

Theorem 2.1. (See [8, Theorem 5.1.7].) *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X .*

(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \quad (2)$$

(ii) *If X is a star set for μ then $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ ($\mathbf{x} \in \mathbb{R}^k$).*

Writing $H = G - X$, we see that the columns \mathbf{b}_u ($u \in X$) of B are the characteristic vectors of the H -neighbourhoods $\Delta_H(u)$ ($u \in X$). Thus G is determined by μ , a star complement H for μ , and the H -neighbourhoods $\Delta_H(u)$ ($u \in X$). From Eq. (2) we have

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

From Eq. (3) we deduce:

Lemma 2.2. (See [8, Proposition 5.1.4].) *Let X be a star set for μ in G , and let $H = G - X$.*

(i) *If $\mu \neq 0$ then $V(H)$ is a dominating set in G .*

(ii) *If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set in G , that is, the H -neighbourhoods $\Delta_H(u)$ ($u \in X$) are non-empty and distinct.*

We shall also need the following observation, which follows from the fact the multiplicity of an eigenvalue changes by 1 at most when a vertex is deleted (cf. [8, Corollary 1.3.12]).

Lemma 2.3. *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

The next lemma extends the result of [2] mentioned in Section 1.

Lemma 2.4. (See [13, Proposition 1.5(ii)] *Let G be a graph with X as a star set for μ , and let $H = G - X$. If $\mu \notin \{-1, 0\}$ and $|\cup_{i \in X} \Delta_H(i)| = d$, then $|X| \leq \binom{d+1}{2}$.*

Recall that μ is said to be a *main* eigenvalue of G if G has a μ -eigenvector not orthogonal to the all-1 vector in \mathbb{R}^n . From the description of $\mathcal{E}(\mu)$ in Theorem 2.1(ii), we have:

Lemma 2.5. (See [6, Proposition 0.3].) *The eigenvalue μ is non-main if and only if $\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{j} = -1$ for all $u \in X$.*

Recall that a (κ, τ) -regular set in a graph G is a set S of vertices such that (i) S induces a κ -regular subgraph, and (ii) every vertex not in S has τ neighbours in S . The following observation is implicit in [12, Proposition 1.5]; an alternative argument is given in [1, Theorem 3.2].

Lemma 2.6. *Let G be a graph with a κ -regular star complement H for the eigenvalue μ . Then μ is a non-main eigenvalue of G if and only if $V(H)$ is (κ, τ) -regular with $\tau = \kappa - \mu$.*

Proof. Let $H = G - X$. By Lemma 2.5, μ is a non-main eigenvalue if and only if $\mathbf{b}_u^T(\mu - \kappa)^{-1}\mathbf{j} = -1$ for all $u \in X$, equivalently $|\Delta_H(u)| = \kappa - \mu$ for all $u \in X$. \square

Several structural conditions sufficient to ensure that $\kappa'(G) = \delta(G)$ may be found in the survey paper [9]. An early example is the following result, due to Chartrand [3].

Lemma 2.7. *If the graph G has order $n \leq 2\delta(G) + 1$ then $\kappa'(G) = \delta(G)$.*

Finally we note a recent result of Cioabă [4] relevant to our consideration of regular star complements.

Theorem 2.8. (See [4, Theorem 1.3].) *Let k, s be integers such that $s \geq k \geq 2$, and let H be an s -regular graph of order t . If $\lambda_2(H) \leq s - \frac{(k-1)t}{(s+1)(t-s-1)}$ then $\kappa'(H) \geq k$.*

3 Edge-connectivity

Theorem 3.1. *Let $k \in \mathbb{N}$, and let G be a graph with H as a star complement for a non-zero eigenvalue μ . If $\kappa'(H) \geq k$ and $\delta(G) \geq k$ then $\kappa'(G) \geq k$.*

Proof. The result holds for $k = 1$ because $V(H)$ is a dominating set by Lemma 2.2(i). Accordingly we assume that $k > 1$ and suppose by way of contradiction that G has a cutset E with $|E| \leq k - 1$. Let $V(G) = U \dot{\cup} V$, where each edge in E joins U to V . If $V(H)$ meets both U and V then $\kappa'(H) < k$, contrary to assumption, and so without loss of generality, $V(H) \subseteq U$. Let u_1, \dots, u_p be the vertices of U adjacent to V , and let $V = \{v_1, \dots, v_q\}$. Since $V(H)$ is a dominating set, each vertex of V is adjacent to U . Now we use an argument of Plesnik [10, Theorem 6].

Let $s_i = |\Delta_U(v_i)|$ ($i = 1, \dots, q$). Since v_i is adjacent to at most $q - 1$ vertices of V , we have

$$s_i + q - 1 \geq \deg(v_i) \geq k \geq |E| + 1 \quad (i = 1, \dots, q).$$

Hence $\sum_{i=1}^q s_i + q(q - 1) \geq q(|E| + 1)$, that is, $|E| + q(q - 1) \geq q(|E| + 1)$. Hence $q(q - 1) \geq (q - 1)|E| + q$. Since $|E| \geq q$, this is a contradiction. \square

Corollary 3.2. *Let G be a graph with H as a star complement for a non-zero eigenvalue μ .*

(i) *If $\kappa'(H) \geq \delta(G)$ then $\kappa'(G) = \delta(G)$.*

(ii) *If G is regular and $\kappa'(H) \geq k$ then $\kappa'(G) \geq k$.*

Proof. (i) Applying Theorem 3.1 with $k = \delta(G)$, we have $\kappa'(G) \geq \delta(G)$. Always $\kappa'(G) \leq \delta(G)$, and so the result follows.

(ii) Here Theorem 3.1 applies because $\delta(G) \geq \delta(H) \geq \kappa'(H) \geq k$. \square

We see that $\kappa'(G)$ is controlled by $\kappa'(H)$ and $\delta(G)$; explicitly, we have $\min\{\kappa'(H), \delta(G)\} \leq \kappa'(G) \leq \delta(G)$. Moreover, for each $k \in \mathbb{N}$, there exists

a least non-negative integer $g(k) \leq k$ such that the relation (1) holds. The following example shows that $g(k) = k$ for all $k > 1$. (We know already that $g(1) = 0$, because G is connected whenever H is connected.)

Example 3.3. For $k \geq 2$, let G_k be the graph obtained from a $(k+1)$ -clique H_k by adding a vertex of degree $k-1$, and let $\mu = \lambda_1(G_k)$. Since G_k is connected, we have $\mu > \lambda_1(H_k)$, and so H_k is a star complement for μ in G_k . Now $\kappa'(G_k) = k-1 = \delta(G_k)$, while $\kappa'(H_k) = k$. Hence $g(k) \geq k$. \square

In what follows, we investigate the situations in which a strict inequality holds in the hypotheses of Theorem 3.1, that is either (a) $\kappa'(H) \geq k$ and $\delta(G) \geq k+1$ (see Corollary 3.5), or (b) $\kappa'(H) \geq k+1$ and $\delta(G) \geq k$ (see Corollary 3.7).

Proposition 3.4. *Let G be a graph with H as a star complement for a non-zero eigenvalue μ . If $\kappa'(H) = \kappa'(G)$ and G has a minimum cutset $E \not\subseteq E(H)$ then $\kappa'(G) = \delta(G)$.*

Proof. We define U and V as in Theorem 3.1. Again we may take $V(H) \subseteq U$, for otherwise H can be disconnected by removing the edges in $E \cap E(H)$. If $\kappa'(G) < \delta(G)$ then we have

$$s_i + q - 1 \geq \deg(v_i) \geq \delta(G) \geq |E| + 1 \quad (i = 1, \dots, q),$$

and we obtain a contradiction as before. \square

Corollary 3.5 *Let G be a graph with H as a star complement for a non-zero eigenvalue μ . If $\kappa'(H) \geq k$ and $\delta(G) \geq k+1$ then either (a) $\kappa'(G) \geq k+1$ or (b) $\kappa'(H) = \kappa'(G) = k$ and every minimum cutset of G lies in $E(H)$.*

Proof. By Theorem 3.1, we have $\kappa'(G) \geq k$; moreover, (a) holds if $\kappa'(H) \geq k+1$. If $\kappa'(G) = k$ then $\kappa'(H) = k$, and (b) holds by Proposition 3.4. \square

We remark in passing that if E is a non-trivial cutset of G in $E(H)$ then the multiplicity of μ is subject to an upper bound which improves that given in [2, Theorem 2.3]. This last result says that if $H = G - X$ of order $t > 4$ then $|X| \leq \binom{t}{2}$. On the other hand, if $V(H) = V_1 \dot{\cup} V_2$, where each edge in E joins V_1 to V_2 , let $|V_i| = t_i$, $X_i = \{u \in X : \Delta_H(u) \subseteq V_i\}$ ($i = 1, 2$). Then $t = t_1 + t_2$, where $t_1 \geq 2$ and $t_2 \geq 2$ because E is non-trivial. Moreover, $X = X_1 \dot{\cup} X_2$ and by Lemma 2.3 we may apply Lemma 2.4 to $G - X_1$, $G - X_2$ to deduce that $|X| \leq \binom{t_1+1}{2} + \binom{t_2+1}{2}$. Now when $t_1 \geq 2$ and $t_2 \geq 2$ we have $\binom{t_1+1}{2} + \binom{t_2+1}{2} \leq \binom{t_1+t_2}{2}$, with strict inequality unless $t_1 = t_2 = 2$.

We say that a set E of edges in G is a k -clique matching if E consists of independent edges $u_i v_i$ ($i = 1, \dots, k$) such that the vertices v_1, \dots, v_k induce a clique which is a component of $G - E$. (Note that, in a connected graph, a 1-clique matching is a trivial minimum cutset consisting of a pendant edge.) For a vertex v of G we write $E(v)$ for the set of edges containing v .

Proposition 3.6. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. If $\kappa'(H) \geq \kappa'(G) = k$ and E is a minimum cutset of G then one of the following holds:*

- (a) $\kappa'(H) = k$ and $E \subseteq E(H)$;
- (b) $E = E(v)$ for some $v \notin V(H)$;
- (c) $E \cap E(H) = \emptyset$ and E is a k -clique matching.

Proof. Note first that if $E \subseteq E(H)$ then $\kappa'(H) = k$, and (a) holds. Now suppose that $E \not\subseteq E(H)$. Since $\kappa'(H) \geq k$ we have $E \cap E(H) = \emptyset$, and we may define U, V as before, with $V(H) \subseteq U$. In the notation of Theorem 3.1 we have

$$s_i + q - 1 \geq \deg(v_i) \geq \delta(G) \geq \kappa'(G) = |E| \quad (i = 1, \dots, q), \quad (4)$$

whence $\sum_{i=1}^q s_i + q(q-1) \geq q|E|$, that is,

$$q(q-1) \geq (q-1)|E|.$$

If $q = 1$, we have case (b). If $q > 1$ then $q \geq |E|$ and necessarily $q = |E| = k$. In this situation, $\sum_{i=1}^q s_i = q$ and so all s_i are equal to 1. By Lemma 2.2(ii), the edges in E are independent. Since equality holds throughout Eq. (4), the vertices v_1, \dots, v_q induce a clique, and so E is a k -clique matching. \square

Corollary 3.7. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. If $\kappa'(H) \geq k+1$ and $\delta(G) \geq k$ then either (a) $\kappa'(G) \geq k+1$ or (b) $\kappa'(G) = k$ and every nontrivial minimum cutset of G is a k -clique matching.*

Proof. We first apply Theorem 3.1: if $\delta(G) \geq k+1$ then $\kappa'(G) \geq k+1$, and if $\delta(G) = k$ then $\kappa'(G) = k$. In the latter case, (b) follows from Proposition 3.6 because a cutset of size k cannot lie in $E(H)$. \square

Next we investigate case (c) of Proposition 3.6: we establish a connection between k and the spectrum of H when $\kappa'(H) \geq k$ and G has a k -clique matching $E \not\subseteq E(H)$. Recall that the condition $\kappa'(H) \geq k$ ensures that $E \cap E(H) = \emptyset$.

Theorem 3.8. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$, with $\kappa'(H) \geq \kappa'(G) = k > 1$. Suppose that G has a k -clique matching $E \not\subseteq E(H)$, and let $\nu_1, \nu_2, \dots, \nu_t$ be the eigenvalues of H in non-increasing order.*

(i) *If $\mu > -1$ then there exists a smallest h such that $\nu_h < \mu$. In this case, $t \geq h + k - 2$ and $\nu_{h+k-2} \geq \mu - \frac{1}{\mu+1}$.*

(ii) *If $\mu < -1$ then there exists a largest m such that $\nu_m > \mu$. In this case, $m \geq k$, $\nu_{m-k+1} \leq \mu - \frac{1}{\mu+1}$ and $\nu_m \leq \mu + \frac{1}{k-\mu-1}$.*

Proof. As before, we let $V(G) = U \dot{\cup} V$, where each edge in E joins U to V and $V(H) \subseteq U$. We apply Theorem 2.1 to the graph obtained from G by deleting the vertices in $U \setminus V(H)$; by Lemma 2.3, H remains a star complement for μ . We have

$$(\mu + 1)I - J = B^\top(\mu I - C)^{-1}B \quad (5)$$

where J is the all-1 matrix of size $k \times k$, C is the adjacency matrix of H and without loss of generality, $B^\top = (I|O)$. Since $B^\top B = I$, the eigenvalues of $(\mu + 1)I - J$ interlace those of $(\mu I - C)^{-1}$ in accordance with [8, Theorem 1.3.11].

(i) *The case $\mu > -1$.* If $\nu_t > \mu$ then all eigenvalues of H exceed -1 and so $H = \bar{K}_t$, a contradiction since $\kappa'(H) \geq 2$. Hence there is a smallest h

such that $\nu_h < \mu$, and $(\mu I - C)^{-1}$ has $t - h + 1$ positive eigenvalues, namely $(\mu - \nu_h)^{-1}, (\mu - \nu_{h+1})^{-1}, \dots, (\mu - \nu_t)^{-1}$ in non-increasing order. Now the $k - 1$ largest eigenvalues of $(\mu + 1)I - J$ are all equal to $\mu + 1$. By interlacing, $\lambda_i((\mu I - C)^{-1}) \geq \mu + 1 > 0$ ($i = 1, \dots, k - 1$), and so $t - h + 1 \geq k - 1$; moreover $\mu + 1 \leq (\mu - \nu_{h+k-2})^{-1}$. equivalently, $\nu_{h+k-2} \geq \mu - \frac{1}{\mu+1}$.

(ii) *The case $\mu < -1$.* Since $\mu < 0$, we have $\nu_1 > \mu$ and so there is a largest m such that $\nu_m > \mu$. The negative eigenvalues of $(\mu I - C)^{-1}$ are $(\mu - \nu_m)^{-1}, (\mu - \nu_{m-1})^{-1}, \dots, (\mu - \nu_1)^{-1}$ in non-decreasing order, while $(\mu + 1)I - J$ has k negative eigenvalues. Hence $m \geq k$. By interlacing, $\lambda_{t-k+1}((\mu I - C)^{-1}) \leq \lambda_1((\mu + 1)I - J)$, that is, $(\mu - \nu_{m-k+1})^{-1} \leq \mu + 1$, equivalently $\nu_{m-k+1} \leq \mu - \frac{1}{\mu+1}$. Also by interlacing, $\lambda_t((\mu I - C)^{-1}) \leq \lambda_k((\mu + 1)I - J)$, that is, $(\mu - \nu_m)^{-1} \leq \mu + 1 - k$, equivalently $\nu_m \leq \mu + \frac{1}{k-\mu-1}$. \square

Corollary 3.9. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$, with $\kappa'(H) \geq \kappa'(G) = k > 1$. Suppose that G has a k -clique matching $E \not\subseteq E(H)$.*

(i) *If $\mu > -1$ then $k - 1$ is bounded above by the number of eigenvalues of H in the interval $[\mu - \frac{1}{\mu+1}, \mu)$.*

(ii) *If $\mu < -1$ then k is bounded above by the number of eigenvalues of H in the interval $(\mu, \mu - \frac{1}{\mu+1}]$.*

We see that (in the situation of Corollary 3.9) for any $\mu \notin \{-1, 0\}$, we have $\kappa'(G) \leq 1 + e_H(\mu)$, where $e_H(\mu)$ is the number of eigenvalues of H between μ and $\mu - \frac{1}{\mu+1}$ inclusive. We shall give an example in which this bound is attained for $\kappa'(G) = 3$. The example arises in the context of the following result, where we apply Theorem 3.8 in the case that H is regular and μ is non-main.

Theorem 3.10. *Let G be a graph with the s -regular graph H as a star complement for the non-main eigenvalue μ . Let $\kappa'(G) = k$, where $s \geq k > 1$ and let $\nu_1, \nu_2, \dots, \nu_t$ be the eigenvalues of H in non-increasing order. Suppose that G has a k -clique matching $E \not\subseteq E(H)$. Then $\mu = s - 1$ and the following hold.*

(i) *If $\nu_2 \leq s - \frac{2(k-1)}{s+1}$ then $\kappa'(H) \geq k$.*

(ii) *If $\kappa'(H) \geq k$, $l > 2$ and $\nu_l < s - 1 - \frac{1}{s}$ then $k \leq l - 1$.*

Proof. We use the notation of Theorem 3.8. Let G^* be the graph obtained from G by deleting the vertices in $U \setminus V(H)$. Note that μ remains a non-main eigenvalue of G^* by Lemma 2.5. Since $V(H)$ is an $(s, 1)$ -regular set in G^* , we have $\mu = s - 1$ by Lemma 2.6.

Assertion (i), due to Cioabă [4], holds whether or not μ is a main eigenvalue. Indeed, if H has order $t \leq 2s + 1$ then $\kappa'(H) = s \geq k$ by Lemma 2.7, while if $t \geq 2s + 2$ then $\nu_2 \leq s - \frac{2(k-1)}{s+1} \leq s - \frac{(k-1)t}{(s+1)(t-s-1)}$ and we have $\kappa'(H) \geq k$ by Theorem 2.8.

For (ii) we note that $\nu_1 = s$, and so if $\nu_l < s - 1 - \frac{1}{s}$ then H has at most $l - 2$ eigenvalues in the interval $[s - 1 - \frac{1}{s}, s - 1)$. By Corollary 3.9(i), we have $k - 1 \leq l - 2$. \square

We illustrate Theorem 3.10 with the following example, found experimentally using the computer package GRAPH [5].

Example 3.11. Let H be the 3-regular graph of order 10 which appears as the second graph in the list of exceptional regular graphs given in [7, Appendix A3.3]. Let G be the graph obtained from H by adding a 3-clique and the 3-clique matching E shown in Fig. 1, where the edges in E join white vertices to black. Note that $\kappa'(H) = 3$, either directly or by Theorem 3.10(i), and $\kappa'(G) = 3$ by construction. The spectrum of G is

$$3.2731, 2^{(3)}, 0.8596, 0^{(2)}, -1^{(2)}, -2^{(3)}, -2.1326,$$

where non-integer eigenvalues are given to four decimal places. The spectrum of H is

$$3, 1.8794^{(2)}, 1, -0.3473^{(2)}, -1.5321^{(2)}, -2^{(2)},$$

and so H is a star complement for 2. By Lemma 2.6, 2 is a non-main eigenvalue of G . In the notation of Theorem 3.10, we have $1 = \nu_4 < s - 1 - \frac{1}{s} = \frac{5}{3}$, and so the bound in Theorem 3.10(ii) is sharp for $l = 4$. Similarly, the bound in Corollary 3.9(i) is sharp for $\mu = 2$. \square

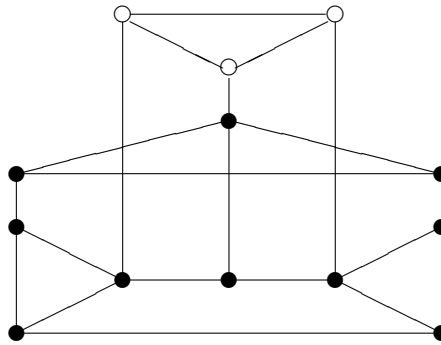


Fig. 1. The graph of Example 3.12.

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